Combinatorial stochastic processes and the reconstruction of macroeconomics

Enrico Scalas

DISTA, East Piedmont University, Italy

3–5 September 2009,
Háskólinn í Reykjavík, Reykjavík, Iceland

Outline

Outline

Distributional problems in Economics

Distributional problems appear in the following frameworks

- growth (and its dependence on firms' sizes);
- allocation of resources (in particular the distribution of wealth).

This presentation emphasizes the positive (rather than the normative) aspect of distributional problems.

General policy problems

«... when, in economic reasoning, the social wealth distribution is assumed "given", this means that the existing distribution is accepted, without evaluating whether it is good or bad, acceptable or unacceptable ... this must be explicitly done further clarifying that the conclusions are conditioned on the acceptabiliy of the distributional set-up.»
F. Caffè (1978), *Lezioni di politica economica*, when introducing the paper by A. Bergson *A reformulation of certain aspects of welfare economics*, Q. J. Econ. 52:310-334.

Two main theoretical approaches

Two main probabilistic methods were used to derive/justify observed/empirical distributions:

- the statistical equilibrium method. According to this approach, the time evolution of an economic system is represented by an aperiodic, irreducible Markov chain and the distribution of relevant quantities is given by the invariant distribution of the Markov chain.
- The diffusive (possibly non-equilibrium) method. According to this approach, the time evolution of an economic system is represented by a random walk.

The random walk method in a nutshell

$x(t) = \log(s(t))$, where $s(t)$ is a "size".

$$x(t) = x(0) + \sum_{m=0}^{t-1} \xi(m) \tag{1}$$

where $\xi(m)$ are independent and identically distributed random variables with probability density $\lambda(\xi)$.

$$p(x,t) = p_0(x) * [\lambda(x)]^{*t}, \tag{2}$$

where $p_0(x)$ is the distribution of $x(0)$ and $*$ represent the convolution operator. For large t, one has that $p(x,t) \simeq N(t\mathbb{E}(\xi), t\mathrm{Var}(\xi))$; for fixed t, x is normal and s is log-normal, but $\lim_{t\to\infty} p(x,t)$ does not exist!

Invitation to further reading I

1. J. Aitchison and J.A.C. Brown, The Lognormal Distribution, Cambridge University Press, Cambridge, UK (1957).
2. J. Angle, The Surplus Theory of Social Stratification and the Size Distribution of Personal Wealth, Soc. Forces 65 (1986), pp. 293-326.
3. E. Bennati, Un metodo di simulazione statistica per l'analisi della distribuzione del reddito, Riv. Int. Sci. Econ. Com. 35 (1988), pp. 735-756.
4. E. Bennati, Il metodo di Montecarlo nell'analisi economica, Rass. Lavori dell'ISCO, Anno X (4) (1993), pp. 31-79.
5. D.G. Champernowne and F.A. Cowell, Economic Inequality and Income Distribution, Cambridge University Press, Cambridge, UK (1999).

Invitation to further reading II

1. A. Drăgulescu and V.M. Yakovenko, Statistical mechanics of money, Eur. Phys. J. B 17 (2000), pp. 723-729.
2. M. Levy, Market efficiency, the Pareto wealth distribution and the Lévy distribution of stock returns. In: L. Blume and S.N. Durlauf, Editors, The Economy as an Evolving Complex System 3, Oxford University Press (2005).
3. T. Lux, Emergent statistical wealth distributions in simple monetary exchange models: a critical review. In: A. Chatterjee, S. Yarlagadda and B.K. Chakrabarti, Editors, Econophysics of Wealth Distribution, Springer, Berlin (2005).
4. J. Steindl, Random Processes and the Growth of Firms - A Study of the Pareto Law, Charles Griffin and Company, London (1965).

Co-workers

From now on, the focus will be on the statistical equilibrium method! The work described below was performed in collaboration with

- Stefania Donadio (Genoa)
- Ubaldo Garibaldi (Genoa)
- Paolo Viarengo (Naples)

Random variables

General framework for agent-based models: there are n objects/individuals to be allocated into g categories/classes;

- a vector $\mathbf{X}^{(n)} = (X_1 = x_1, \ldots, X_n = x_n)$ characterizes an *individual* description. $X_i = j$ means that the i-th agent belongs to the j-th category. The total number of individual descriptions is g^n;
- a vector $\mathbf{n} = (n_1, \ldots, n_g)$ characterizes a single *occupation* state, where $n_j = \#\{i : X_i = j\}$. Note that $\sum_{j=1}^{g} n_j = n$. The total number of occupation states is

$$\binom{n+g-1}{n} = \binom{n+g-1}{g-1}; \tag{3}$$

- a vector $\mathbf{z} = (z_0, z_1, \ldots, z_n)$ characterizes a single *partition* state, where $z_k = \#\{j : n_j = k\}$. Note that $\sum_{k=0}^{n} z_k = g$ and $\sum_{k=0}^{n} k z_k = n$. There is no closed-form expression for the number of partitions.

Example (3 objects in 2 categories)

- eight individual descriptions: $(1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(2,1,1)$, $(2,2,1)$, $(2,1,2)$, $(1,2,2)$, $(2,2,2)$.
- four occupation vectors $(3,0)$ corresponding to $(1,1,1)$; $(2,1)$ corresponding to $(1,1,2)$, $(1,2,1)$, and $(2,1,1)$; $(1,2)$ corresponding to $(2,2,1)$, $(2,1,2)$, and $(1,2,2)$; $(0,3)$ corresponding to $(2,2,2)$.
- two partition vectors $(1,0,0,1)$ corresponding to $(3,0)$ and $(0,3)$; $(0,1,1,0)$ corresponding to $(1,2)$ and $(2,1)$.

Note that:

- for each occupation vector $\mathbf{n} = (n_1, \ldots, n_g)$ there are

$$\frac{n!}{\prod_{i=1}^{g} n_i!} \tag{4}$$

corresponding individual descriptions;

- for each partition vector $\mathbf{z} = (z_0, z_1, \ldots, z_n)$ there are

$$\frac{g!}{\prod_{i=0}^{n} z_i!} \tag{5}$$

corresponding occupation vectors.

Stochastic processes

The sequence of individual random variables $X_1, \ldots, X_n$ is an n-step stochastic process. It is completely determined by the knowledge of all the finite dimensional distributions of the kind

$$p(x_1, \ldots, x_m) = \mathbb{P}(X_1 = x_1, \ldots, X_m = x_m) \tag{6}$$

where $1 \leq m \leq n$. The finite dimensional distributions are subject to Kolmogorov's compatibility conditions

$$p(x_1, \ldots, x_m) = p(x_{i_1}, \ldots, x_{i_m}), \tag{7}$$

where $i_1, \ldots, i_m$ is any of the $m!$ permutations of the indices, and

$$p(x_1, \ldots, x_{m-1}) = \sum_{x_m=1}^{g} p(x_1, \ldots, x_{m-1}, x_m). \tag{8}$$

Predictive probabilities

Finite dimensional distributions can be conveniently characterized in terms of *predictive* probabilities. Indeed, as a consequence of the multiplication theorem (and of Bayes' theorem), one has

$$\mathbb{P}(X_1 = x_1, \ldots, X_m = x_m) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2 | X_1 = x_1) \cdots \cdots \mathbb{P}(X_m = x_m | X_1 = x_1, \ldots, X_{m-1} = x_{m-1}) \quad (9)$$

and Kolmogorov's compatibility conditions are automatically satisfied (see also Ionescu Tulcea's theorem).

Exchangeable processes

An exchangeable process is characterized by additional symmetry conditions on the finite dimensional distributions

$$\mathbb{P}(X_1 = x_1, \ldots, X_m = x_m) = \mathbb{P}(X_{i_1} = x_1, \ldots, X_{i_m} = x_m). \quad (10)$$

where $i_1, \ldots, i_m$ is any of the $m!$ permutations of the indices, Note that condition (10) differs from condition (7). Indeed, the latter can be written as

$$\mathbb{P}(X_1 = x_1, \ldots, X_m = x_m) = \mathbb{P}(X_{i_1} = x_{i_1}, \ldots, X_{i_m} = x_{i_m}) \quad (11)$$

for any permutation of the indices.

For an exchangeable process, the probability of an individual sequence $\mathbf{X}^{(m)} = (X_1 = x_1, \ldots, X_m = x_m)$ only depends on the occupation vector of the sequence $\mathbf{m} = (m_1, \ldots, m_g)$ with $\sum_{i=1}^{g} m_i = m$. This leads to

$$\mathbb{P}(\mathbf{X}^{(m)}) = \left(\frac{m!}{\prod_{i=1}^{g} m_i!} \right)^{-1} \mathbb{P}(\mathbf{m}) \tag{12}$$

as a consequence of (4).

The Pólya process

The Pólya process is an exchangeable process characterized by the predictive probability

$$\mathbb{P}(X_{m+1} = j | X_1 = x_1, \ldots, X_m = x_m) = \frac{\alpha_j + m_j}{\alpha + m}, \qquad (13)$$

where m_j is the number of times in which category j has been observed up to step j, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)$ is a vector of parameters and $\alpha = \sum_{i=1}^{g} \alpha_i$. If the new parameters $p_j = \alpha_j / \alpha$ are introduced, equation (13) becomes

$$\mathbb{P}(X_m = j | X_1 = x_1, \ldots, X_m = x_m) = \frac{\alpha p_j + m_j}{\alpha + m}. \qquad (14)$$

$p_j = \mathbb{P}(X_1 = j)$ is the *a priori* probability of category j and (14) is nothing else than a linear mixture between *a priori* probabilities and the observed frequencies.

As a consequence of (13), and of exchangeability (see (12)), one gets the following finite dimensional distributions

$$\mathbb{P}(\mathbf{X}^{(m)}) = \left(\frac{m!}{\prod_{i=1}^{g} m_i!} \right)^{-1} \mathrm{Polya}(\mathbf{m}|m;\boldsymbol{\alpha}) \tag{15}$$

where the multivariate generalized Pólya sampling distribution is given by

$$\mathrm{Polya}(\mathbf{m}|m;\boldsymbol{\alpha}) = \frac{m!}{\alpha^{[m]}} \prod_{i=1}^{g} \frac{\alpha_i^{[m_i]}}{m_i!} \tag{16}$$

where $x^{[n]} = x(x+1)\cdots(x+n-1)$ is the rising factorial.

The Pólya process encompasses the following remarkable cases:

- the multivariate hypergeometric process for integer $\alpha_j < 0$, $\forall j \in \{1, 2, \ldots, g\}$. In this case $|\alpha_j|$ represents the initial number of marbles of colour j in an urn from which they are randomly drawn without replacement; this process is not extendible to infinity and ends after n steps;
- the multinomial process in the limit $|\alpha| \to \infty$ and $|\alpha_j| \to \infty$, with $p_j = \alpha_j/\alpha$ constant. In this case p_j represents the probability of drawing a marble of colour j with replacement from and urn;
- the Pólya urn process for integer $\alpha_j > 0$, $\forall j \in \{1, 2, \ldots, g\}$. In this case α_j is the initial number of marbles of colour j in an urn. They are randomly drawn and replaced with another ball of the same kind.

Outline

Markov chains

Recall the general framework for agent-based models:

- there are n objects to be allocated into g categories;
- a vector $\mathbf{n} = (n_1, \dots, n_g)$ characterizes a single *occupation* state. The total number of occupation states is

$$\binom{n+g-1}{n} = \binom{n+g-1}{g-1}; \tag{17}$$

Now add the following assumption:

- transitions between occupation states, $\mathbf{n} \to \mathbf{n}'$ follow a homogeneous Markov dynamics with transition probablity $P(\mathbf{n}'|\mathbf{n})$.

Master equation

All homogenous Markov chains obey the following *master equation*

$$\pi(\mathbf{n}, t+1) - \pi(\mathbf{n}, t) = \sum_{\mathbf{n}' \neq \mathbf{n}} [P(\mathbf{n}|\mathbf{n}')\pi(\mathbf{n}', t) - P(\mathbf{n}'|\mathbf{n})\pi(\mathbf{n}, t)]. \quad (18)$$

Detailed balance

If *detailed balance* is satisfied, defined as

$$P(\mathbf{n}|\mathbf{n}')\pi(\mathbf{n}', t) = P(\mathbf{n}'|\mathbf{n})\pi(\mathbf{n}, t), \tag{19}$$

then one gets the stationary distribution

$$\pi(\mathbf{n}, t+1) = \pi(\mathbf{n}, t) = \pi(\mathbf{n}). \tag{20}$$

Useful definitions and theorem I

- A Markov chain satisfying detailed balance is said *reversible* with respect to the distribution $\pi(\mathbf{n})$.
- A Markov chain is *irreducible* or *ergodic* if all the states can be reached with finite probability.
- If $\mathbf{n}$ is a state of a Markov chain such that $P^t(\mathbf{n}|\mathbf{n}) > 0$ for some $n \geq 1$, its *period* $d_{\mathbf{n}}$ is defined as the greatest common divisor of the set $\{t \geq 1 : P^t(\mathbf{n}|\mathbf{n}) > 0\}$. For two states $\mathbf{n}$ and $\mathbf{n}'$ leading to each other, $d_{\mathbf{n}} = d_{\mathbf{n}'}$. States in an irreducible Markov chain have a common period d.
- The chain is called *periodic* of period d if $d > 1$ and *aperiodic* if $d = 1$.

Useful definitions and theorem II

Theorem

If a Markov chain is irreducible, aperiodic and reversible with respect to $\pi(\mathbf{n})$, then $\pi(\mathbf{n})$ is the unique stationary distribution of the chain and is also an equilibrium distribution, this last statement meaning that

$$\lim_{t \to \infty} P^t(\mathbf{n}|\mathbf{n}') = \pi(\mathbf{n}), \tag{21}$$

independent of the initial state $\mathbf{n}'$.

Equation (21) shows the meaning of *statistical equilibrium*. It is a *dynamical* equilibrium, where objects always change category, but the probability of any occupation state is governed by the invariant distribution $\pi(\mathbf{n})$.

Outline

Aoki-Yoshikawa model: Definition

The AYM consists of a closed economy

- with g economic sectors and N workers;
- occupation vector $\mathbf{n} := (n_1, n_2, \ldots, n_g)$, where n_i is the number of workers in sector i;
- each sector has productivity c_i with $c_1 < c_2 < \ldots < c_n$;
- the total product of sector i is $Y_i = c_i n_i$;
- the total product of the economy (GDP) is $Y = \sum_{i=1}^{g} Y_i = \sum_{i=1}^{g} c_i n_i$;
- $N = \sum_{i=1}^{g} n_i$;
- exogenously given demand D; at equilibrium $Y = \sum_{i=1}^{g} c_i n_i = D$.

Equilibrium distribution

Problem: maximize the equilibrium distribution

$$\pi(\mathbf{n}) = C\frac{N!}{\prod_{i=1}^{g} n_i!}; \quad (22)$$

subject to constraints

$$N = \sum_{i=1}^{g} n_i \quad (23)$$

and

$$D = \sum_{i=1}^{g} c_i n_i \quad (24)$$

in order to find the most probable occupation vector.

Maximum entropy

Solution (Boltzmann, 1877): minimize $\Pi_{i=1}^{g} n_i!$ subject to constraints. Using Stirling's approximation for the factorial and Lagrange multipliers, one can minimize

$$L(\mathbf{n}) = \sum_{i=1}^{g} n_i(\log n_i - 1) + \nu \left(N - \sum_{i=1}^{g} n_i \right) + \beta \left(D - \sum_{i=1}^{g} c_i n_i \right)$$

leading to

$$n_i = \exp(\nu + \beta c_i) = \exp(\nu) \exp(\beta c_i), \tag{25}$$

where α and β can be obtained from the constraints in equations (23) and (24).

A remarkable case

If $c_i = ic_1$ with $i = 1, 2, \ldots, g$, one gets for $Nc_1/D << 1$

$$n_i \simeq \frac{N^2 c_1}{D} \exp\left(-\frac{Nc_1 i}{D}\right). \tag{26}$$

Then, one can define the *occupation frequencies* n_i/N that sum up to 1 and consider them as probabilities. They follow the exponential distribution

$$\frac{n_i}{N} \simeq \frac{Nc_1}{D} \exp\left(-\frac{Nc_1 i}{D}\right). \tag{27}$$

However, equations (25) or (27) *do not* provide the unique equilibrium distribution compatible with AYM.

Unary moves for AYM

In order to introduce a dynamic version of AYM let us introduce *unary* moves.

- Destruction

$$\mathbf{n}_j := (n_1, \ldots, n_j - 1, \ldots, n_g). \tag{28}$$

- Creation

$$\mathbf{n}^j := (n_1, \ldots, n_j + 1, \ldots, n_g) \tag{29}$$

- Unary move: assume $i < j$, then

$$\mathbf{n}_i^j := (n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_g) \tag{30}$$

Unary moves violate the demand constraint (24).

Binary moves

If $c_i = ic_1$, binary moves fulfill both constraints (23) and (24). A binary move is defined as follows: $\mathbf{n}_{ij}^{mn}$, where $n'_i = n_i - 1$, $n'_j = n_j - 1$, $n'_l = n_l + 1$, and $n'_m = n_m + 1$.
Example: $g = 3$ sectors with respective productivities c_1, $c_2 = 2c_1$, and $c_3 = 3c_1$ and $N = 3$ workers. Suppose that the initial demand is set at the following level $D = 6c_1$. This is fufilled e.g. by an initial state in which all the three workers are in sector 2. Therefore, the initial occupation vector is $\mathbf{n} = (0, 3, 0)$. An allowed binary move leads to state $\mathbf{n}_{22}^{13} = (1, 1, 1)$ where two workers leave sector 2 to jump to sectors 1 and 3, respectively. This state fulfills the demand constraint as $c_1 n_1 + c_2 n_2 + c_3 n_3 = 6c_1$.

Transition probability

After defining binary moves and proper constraints on accessible states, it is possible to define a dynamics on AYM using an appropriate transition probability. A possible choice is:

$$P(\mathbf{n}_{ij}^{lm}|\mathbf{n}) = A_{ij}^{lm}(\mathbf{n}) n_i n_j (1 + \mu n_l)(1 + \mu n_m), \tag{31}$$

where $A_{ij}^{lm}(\mathbf{n})$ is a suitable normalization factor and μ is a model parameter.

Meaning of the transition probability

When a worker leaves sector i, he/she does so with probability

$$P(\mathbf{n}_i|\mathbf{n}) = n_i/N \tag{32}$$

proportional to the number of workers in sector i before the move. When he/she joins sector l, this happens with probability

$$P(\mathbf{n}^l|\mathbf{n}) = (1+\mu n_l)/(g+\mu N). \tag{33}$$

Meaning of μ

A worker will not choose the arrival sector independently from its occupation before the move, but

- he/she will be likely to join more populated sectors if $\mu > 0$,
- or he/she will prefer to stay away from populated sectors if $\mu < 0$,
- or he/she will be equally likely to join any sector if $\mu = 0$.

Remark: if $\mu \geq 0$, there is no restriction in the number of workers who can occupy a sector, whereas for negative values of μ, only situations in which $1/|\mu|$ is integer are allowed and no more than $1/|\mu|$ workers can be allocated in each sector.

Imposing detailed balance

The inverse transition has probability

$$P(\mathbf{n}|\mathbf{n}_{ij}^{mn}) = A_{ij}^{lm}(\mathbf{n}_{ij}^{lm})(n_l+1)(n_m+1)(1+\mu(n_i-1))(1+\mu(n_j-1)). \quad (34)$$

As a consequence of equations (32) and (33) and taking into account the demand constraint, it is possible to show that $A_{ij}^{lm}(\mathbf{n}_{ij}^{lm}) = A_{ij}^{lm}(\mathbf{n})$. Then, the detailed balance condition becomes

$$\frac{\pi(\mathbf{n}_{ij}^{lm})}{\pi(\mathbf{n})} = \frac{n_i n_j(1+\mu n_l)(1+\mu n_m)}{(n_l+1)(n_m+1)(1+\mu(n_i-1))(1+\mu(n_j-1))}. \quad (35)$$

This condition can be studied for different vaues of μ.

Equilibrium distributions I

- If $\mu = 1$ then $\pi(\mathbf{n}_{ij}^{lm})/\pi(\mathbf{n}) = 1$, meaning that $\pi(\mathbf{n})$ is uniform on the set of accessible states;
- if $\mu = -1$ then $\pi(\mathbf{n}_{ij}^{lm})/\pi(\mathbf{n}) = 1$ but only if $n_i = n_j = 1$ and $n_l = n_m = 0$; all the states satisfying an exclusion principle and the demand constraint have the same probability;
- if $\mu = 0$ then

$$\pi(\mathbf{n}_{ij}^{lm})/\pi(\mathbf{n}) = (n_i n_j)/[(n_l + 1)(n_m + 1)], \tag{36}$$

yielding an equilibrium distribution given by

$$\pi(\mathbf{n}) = C(\mathbf{n}_0)\frac{N!}{\prod_{i=1}^{g} n_i!}, \tag{37}$$

where C is a suitable normalization constant depending on the initial state $\mathbf{n}_0$ or, equivalently, on the demand D.

Equilibrium distributions II

All the cases discussed above are instances of the following general equilibrium distribution

$$\pi(\mathbf{n}) = C(\mathbf{n_0}) \frac{(1/\mu)^{[n_i]}}{n_i!}. \tag{38}$$

This is nothing else than a generalized Pólya distribution (16) restricted to all the states reacheable from the initial state $\mathbf{n_0}$ and compatible with the constraints (with $\mu = 1/\alpha$). The conditional maximum problem leads to

$$n_i = \frac{1}{\exp(-\nu)\exp(-\beta c_i) - \mu}. \tag{39}$$

Summary and outlook

- The Pólya sampling distribution plays a fundamental role in the finitary approach to combinatorial stochastic processes;
- the theory of finite Markov chains helps in assessing statistical equilibrium for suitable models of an economy;
- when closed formulae or simulation results are available a comparison with empirical data is possible (this important problem was not discussed here).

For Further Reading I

- E. Scalas, U. Garibaldi, and S. Donadio
  Statistical equilibrium in simple exchange games I
  European Physical Journal B, 2006.
- U. Garibaldi, E. Scalas, and P. Viarengo
  Statistical equilibrium in simple exchange games II
  European Physical Journal B, 2007.
- U. Garibaldi, and E. Scalas
  Finitary probabilistic methods in Econophysics
  Cambridge University Press, in preparation, expected for October 2010.